heat of melting; N, number of grid nodes over space; n, number of grid nodes over time; h, grid step over space;  $\Delta \tau$ , grid step over time;  $\psi$ , solution of the conjugate system; s, number of iteration.

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# INERTIA OF MEASUREMENTS WITH "AUXILIARY-WALL" TYPE HEAT METERS

G. N. Dul'nev, N. V. Pilipenko, and V. A. Kuz'min

The article examines the problem of thermal inertia on the basis of an "auxiliary-wall" type heat meter. It demonstrates the boundaries of applicability of the approximate relationship for calculating non-steady-state heat fluxes.

Heat meters of the "auxiliary wall" type are widely used for measuring heat fluxes, and schematically they are often represented in the form of a plate attached to a semibounded body. It was shown in [1] that for measuring non-steady-state heat fluxes with such heat meters, it is necessary to know the temperature gradient  $\Delta t(\tau)$  on the sensor with known thickness  $\delta$ , and also the criterion  $\varkappa = \frac{\lambda_2}{\lambda_1} \sqrt{\frac{a_1}{a_2}}$ , characterizing the thermophysical properties of the heat meter and the

half-space. The same article also presented the theoretical relationships for determining the flux  $q(\tau)$  in some special cases ( $\kappa = 0, 1, \infty$ ). For determining a variable flux, it is necessary in the general case (with arbitrary values of  $\kappa$ ) to use the relationship (I) whose derivation is presented in Appendix 1:

$$q(\tau) = q'(\tau) + q''(\tau); \tag{1}$$

$$q'(\tau) = \frac{\lambda_1}{1 \pi a_1 \tau} \left\{ 1 + \sum_{n=1}^{\infty} \left[ 1 + \left( \frac{1-\kappa}{\kappa+1} \right)^n \right] \exp\left( - \frac{n^2 A^2}{4\tau} \right) \right\} \Delta t(\tau) = K_0(\tau) \Delta t(\tau);$$
(2)

$$q''(\tau) = \frac{\lambda_1}{1/2\pi a_1} \int_0^{\tau} \frac{\Delta t(\tau) - \Delta t(\xi)}{1/(\tau - \xi)^3} \left\{ 1 - \sum_{n=1}^{\infty} \left[ 1 + \left(\frac{1-\varkappa}{\varkappa + 1}\right)^n \right] \frac{n^2 A^2 - 2(\tau - \xi)}{2(\tau - \xi)} \exp\left[ -\frac{n^2 A^2}{4(\tau - \xi)} \right] \right\} d\xi; \quad (3)$$

$$\varkappa = \frac{\lambda_2}{\lambda_1} \sqrt{\frac{a_1}{a_2}}; \quad A = \frac{\delta}{\sqrt{a_1}}.$$

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However, there are also cases when formulas of the type (1) are difficult to use. For instance, in systems of automatic control of dynamic thermal processes it is preferable to use relationships of the following type [2, 3]

$$q(\tau) = \frac{\lambda_1}{\delta} \Delta t(\tau) , \qquad (4)$$

i.e., with direct proportionality between the flux  $q(\tau)$  and the easily controlled signal  $\Delta t(\tau)$ . In this case, calculations by formulas of type (4) may lead to large errors caused by the thermal inertia of the heat meters. It is accepted practice to characterize the thermal inertia by the parameter  $\eta$  equal to the time interval within which the value of the time-invariable flux, measured with the heat meter, is 0.63 of the flux absorbed by the heat meter at that same instant. It is known that the magnitude of  $\eta$  is greatly affected by the relationship of the thermophysical properties of the heat meter and of the base on which it is situated [3]. It is therefore recommended to produce and place the heat meter in such a way that its thermal inertia is much smaller than the frequency of the measured flux.

In the literature there are recommendations indicating the possibility of using relationships of type (4) in some special cases [1, 2]; on the other hand, we lack substantiated correlations between inertia, frequency, and error of measurement of an arbitrarily changing flux  $q(\tau)$  which could already be used at the stage of designing the heat meters. The parameter of thermal inertia  $\eta$  is usually determined experimentally by suddenly exposing the heat meter to a constant heat flux.

We will examine the analytical method of determining the parameter  $\eta$ . For this, we turn to relationships (1)-(3) and show first that after some time  $\tau = \tau^*$  the coefficient  $K_0$  in (2) assumes the constant value  $K_0 = \text{const.}$  We transform expression (2) for the two boundary cases  $\varkappa = 0$  (heat meter on an insulator) and  $\varkappa = \infty$  (heat meter on an ideal conductor):

$$1)\varkappa = 0$$

2) × = ∞

 $q'_{0}(\tau) = \frac{\lambda_{1}}{1 \pi a_{1} \tau} \left\{ 1 + 2 \sum_{n=1}^{\infty} \exp\left(-\frac{n^{2} A^{2}}{4\tau}\right) \right\} \Delta t(\tau);$ (5)

$$q'_{\infty}(\tau) = \frac{\lambda_1}{\sqrt{\pi a_1 \tau}} \left\{ 1 + \sum_{n=1}^{\infty} \left[ 1 + (-1)^n \right] \exp\left(-\frac{n^2 A^2}{4\tau}\right) \right\} \Delta t(\tau) = \frac{\lambda_1}{\sqrt{\pi a_1 \tau}} \left\{ 1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{n^2 A^2}{\tau}\right) \right\} \Delta t(\tau).$$
(6)

In the theory of elliptical functions, the following expression, obtained on the basis of Poisson's summation formula [4], is used:

$$\sum_{n=-\infty}^{\infty} \exp\left(-tn^2\right) = \sqrt{\frac{\pi}{t}} \exp\left(\frac{n^2\pi^2}{t}\right), \quad t > 0.$$
(7)

If in (7) we put  $t = \delta^2 / 4a_1 \tau$ , we have

$$\sum_{n=-\infty}^{\infty} \exp\left(-\frac{\delta^2 n^2}{4a_1 \tau}\right) = \sqrt{\frac{4\pi a_1 \tau}{\delta^2}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{4\pi^2 n^2 a_1 \tau}{\delta^2}\right).$$
(8)

On account of evenness, the exponent under the sign of summation is

$$1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{\delta^2 n^2}{4a_1 \tau}\right) = \frac{2\sqrt{\pi a_1 \tau}}{\delta} \left[1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{4\pi^2 n^2 a_1 \tau}{\delta^2}\right)\right].$$
(9)

In that case

$$q_{0}'(\tau) = \frac{\lambda_{1}}{\sqrt{\pi a_{1}\tau}} \left\{ 1 + 2 \sum_{n=1}^{\infty} \exp\left(-\frac{n^{2}\delta^{2}}{4a_{1}\tau}\right) \right\} \Delta t(\tau) =$$

$$= \frac{\lambda_{1}}{\sqrt{\pi a_{1}\tau}} \frac{2\sqrt{\pi a_{1}\tau}}{\delta} \left\{ 1 + 2 \sum_{n=1}^{\infty} \exp\left(-\frac{4\pi^{2}n^{2}a_{1}\tau}{\delta^{2}}\right) \right\} \Delta t(\tau) =$$

$$= \frac{2\lambda_{1}}{\delta} \left\{ 1 + 2 \sum_{n=1}^{\infty} \exp\left(-\frac{4\pi^{2}n^{2}}{A^{2}}\tau\right) \right\} \Delta t(\tau) = K_{1}(\tau) \Delta t(\tau).$$
(10)



Fig. 1. Comparison of the specified (curves 1, 3;  $\omega_1 = 0.5 \text{ sec}^{-1}$ ;  $\omega_3 = 0.07 \text{ sec}^{-1}$ ) and the calculated (curves 2, 4) heat flux; q, W/m<sup>2</sup>;  $\tau$ , sec.

In (7) we substitute  $t = \delta^2 / a_1 \tau$ , and after some simple transformations we obtain

$$q'_{\infty}(\tau) = \frac{\lambda_1}{\delta} \left\{ 1 + 2 \sum_{n=1}^{\infty} \exp\left(-\frac{\pi^2 n^2}{A^2} \tau\right) \right\} \Delta t(\tau) = K_2(\tau) \Delta t(\tau).$$
(11)

It follows from expressions (10) and (11) that after some time  $\tau$ , the coefficients  $K_1$  and  $K_2$  assume a constant value  $K_1 = (2\lambda_1/\delta)$  ( $\varkappa = 0$ ),  $K_2 = \lambda_1/\delta$  ( $\varkappa = \infty$ ). On a computer, numerical calculations by Eq. (1) involving actual heat meters type DTP-02 [3] ( $\delta = 1.8 \cdot 10^{-3}$  m;  $\lambda_1 = 0.7$  W/mK;  $a = 0.45 \cdot 10^{-6}$  m<sup>2</sup>/sec), situated on a base ( $\varkappa = 35$ ), were carried out. The results of the experimental investigations showed that the time of establishing  $K_0 = \text{const}$ , equal to  $\tau^* = 3.5$  sec, did not depend on the regularity of the change of the incident flux for the boundary cases  $\varkappa = 0$ ,  $\varkappa = \infty$  1 <  $\tau^* < 4c$ .

Thus, on the basis of analytical and experimental investigations it may be asserted that after a certain time interval  $\tau = \tau^*$ , the coefficients  $K_j = K_0$ ;  $K_1$ ;  $K_2$  in expressions (2), (10), (11) assume a constant value, and this time does not depend on the regularity of change of the incident flux.

From relations (2), (10), (11) we find the time  $\tau = \tau^*$  and take it equal to the inertia parameter  $\eta$ , i.e.,

$$\tau^* = \tau|_{K_i = \text{const}} = \eta. \tag{12}$$

A substantiation of equality (12) may be the above-mentioned analytical investigations of boundary cases, and also the results of a computer experiment which was carried out according to the following procedure:

a) a certain regularity of change in the flux  $q(\tau)$  incident on the heat meter (in particular also q = const) was specified;

b) the general form of the function  $\Delta t(\tau) = f[q(\tau)]$  (see Appendix 2) was established;

c) the numerical values were calculated of  $\Delta t(\tau)$  for an actual heat meter and the conditions of its disposition according to the function (XI);

d) the regularity of the change of  $q(\tau)$  was established by Eq. (4) in which the value  $\Delta t(\tau)$  from the previous point was substituted;

e) the specified (standard) flux was compared with the flux calculated by Eq. (4), and the error of calculating  $\Delta q(r)$  was determined;

f) on the basis of the permissible error, the boundaries of applicability of the relation (4) were established.

Figure 1 presents the results of the investigations for the nonperiodic regularity of change of the heat flux with different frequencies, from which it follows that the calculation error for  $\omega = 0.5 \text{ sec}^{-1}$  exceeds 50% and drops to less than half when the frequency is  $\omega = 0.07 \text{ sec}^{-1}$ .

Thus, when the frequency of the flux incident on the heat meter is known, the expected error of measuring  $q(\tau)$  can be calculated at the design stage.

Figure 2 shows the results of calculating by relation (11) (curve 2) and by relation (4) (curve 1). The value  $\Delta t(\tau)$ , necessary for the calculation, was determined by formula (XI) with the specified constant flux  $q = 10^3 \text{ W/m}^2$ . It can be seen from Fig. 2 that after a certain time interval ( $\tau = 3.5 \text{ sec}$ ), curves 1 and 2 coincide (the divergence does not exceed the



Fig. 2. Values of the heat flux calculated by different relations: 1) Eq. (4); 2) Eq. (3).

error of computer calculation). And in all cases, this time is equal to the parameter  $\eta$  characterizing thermal inertia.

Investigations according to the examined procedure also make it possible to establish the boundaries of applicability of the reactions type (4). For this, it is necessary already at the design stage to calculate the value  $q''(\tau)$  and to compare it with the magnitude of the permissible error. If  $q''(\tau) \leq \Delta q(\tau)$ , then the further determination of the flux  $q(\tau)$  can be carried out by Eq. (4).

In conclusion we want to point out that for determining nonsteady-state heat fluxes with the aid of "auxiliary-wall" type heat meters placed on massive bodies it is necessary, in the general case, to use Eq. (1). If for some reason or other relation (4) has to be used, then the expected error has to be previously estimated.

### **APPENDIX 1**

It was shown in [1] that the mathematical statement of the solved problem has the form (model – plate in a half-space):

$$\frac{\partial t_{i}}{\partial \tau} = a_{i} \left( \frac{\partial^{2} t_{i}}{\partial x^{2}} \right), \quad i = 1; 2, \qquad (I)$$

$$q(\tau) = -\lambda_{1} \frac{\partial t_{1}}{\partial x} \Big|_{x=-\delta}, \qquad (I)$$

$$\lambda_{1} \frac{\partial t_{1}}{\partial x} \Big|_{x=0} = \lambda_{2} \frac{\partial t_{2}}{\partial x} \Big|_{x=0}, \quad t_{1} \Big|_{x=0} = t_{2} \Big|_{x=0}; \qquad (I)$$

$$\frac{\partial t_{2}}{\partial x} \Big|_{x\to\infty} = 0, \text{ or } i_{2} \Big|_{x\to\infty} = \text{const}, \qquad (I)$$

$$t_{i} \Big|_{\tau=0} = t_{c}, \quad i = 1; 2.$$

According to [3], the solution of (I), (II) is

$$q(\tau) = \varphi(\tau) \Delta t(\tau) - \int_{0}^{\tau} [\Delta t(\tau) - \Delta t(\xi)] \frac{\partial \varphi(\tau - \xi)}{\partial \xi} d\xi, \qquad (III)$$

where  $\varphi(\xi)$  is the original of the expression

$$F(s) = \frac{1}{sY_q(s)} = \frac{\lambda_1}{\sqrt{a_1}} \frac{\operatorname{sh} A \sqrt{s} + \varkappa \operatorname{ch} A \sqrt{s}}{\sqrt{s} (\varkappa \operatorname{sh} A \sqrt{s} + \operatorname{ch} A \sqrt{s} - 1)}.$$
 (IV)

To find the original of  $\varphi(\xi)$ , we represent F(s) in the form of the sum of two components:

$$F(s) = \frac{\lambda_1}{\sqrt{a_1}} \frac{\operatorname{sh} A \sqrt{s}}{\sqrt{s}(\varkappa \operatorname{sh} A \sqrt{s} + \operatorname{ch} A \sqrt{s} - 1)} + \frac{\lambda_1}{\sqrt{a_1}} \frac{\varkappa \operatorname{ch} A \sqrt{s}}{\sqrt{s}(\varkappa \operatorname{sh} A \sqrt{s} + \operatorname{ch} A \sqrt{s} - 1)}$$

After expansion of the values of the hyperbolic functions and some transformations, the first component can be written in the form

$$\frac{\lambda_{1}}{\sqrt{a_{1}}} \frac{\operatorname{sh} A \sqrt{s}}{\sqrt{s} (\varkappa \operatorname{sh} A \sqrt{s} + \operatorname{ch} A \sqrt{s} - 1)} = \frac{\lambda_{1}}{\sqrt{a_{1}} s} \frac{1}{\varkappa + 1} \frac{1 + \exp(-A\sqrt{s})}{1 + \frac{\varkappa - 1}{\varkappa + 1} \exp(-A\sqrt{s})}.$$
 (V)

We use the expansion [5]:

$$\frac{1}{1 + m \exp(-A_1 + \overline{s})} = \sum_{n=0}^{\infty} (-1)^n \exp(-nA_1 + \overline{s}) m^n,$$
(VI)
$$m = \frac{\varkappa - 1}{\varkappa + 1}$$

and represent (V) in the form

$$\frac{\lambda_{1}}{1} \frac{\sinh A_{1} \cdot \overline{s}}{1 \cdot \overline{s} \cdot (x \cdot \sinh A_{1} \cdot \overline{s} - 1)} = \frac{\lambda_{1}}{1 \cdot \overline{a_{1}} \cdot \overline{x} + 1} \frac{1 + \exp(-A_{1} \cdot \overline{s})}{1 \cdot \overline{s}} \sum_{n=0}^{\infty} (-1)^{n} \exp(-nA_{1} \cdot \overline{s}) \left(\frac{x - 1}{x + 1}\right)^{n}.$$
(VII)

We also proceed analogously with the second component. After that, the expressions for F(s) can be written as follows:

$$F(s) = \frac{\lambda_{1}}{1 \ a_{1} \ s} \frac{1}{(\varkappa + 1)} \sum_{n=0}^{\infty} \left\{ [1 + \exp(-A_{1} \ s]) + x[1 - \exp(-A_{1} \ s])](-1)^{n} \exp(-nA_{1} \ s] \left(\frac{\varkappa - 1}{\varkappa + 1}\right)^{n} + 2\varkappa \exp(-A_{1} \ s])\exp(-nA_{1} \ s] \sum_{j=0}^{n} (-1)^{j} \left(\frac{\varkappa - 1}{\varkappa + 1}\right)^{j} \right\}.$$
(VIII)

We find the original of (VIII), substitute into (III), and obtain the theoretical relation for determining the sought flux:

We remove (n + 1) to behind the signs of summation and integration, and we also carry out term-by-term composition under the signs of summation. As a result of these operations, expression (IX) is greatly simplified and assumes the form

$$q(\tau) = \frac{\lambda_1}{1 - \pi a_1 \tau} \left\{ 1 + \sum_{n=1}^{\infty} \left[ 1 + \left(\frac{1 - \varkappa}{\varkappa + 1}\right)^n \right] \exp\left(-\frac{n^2 A^2}{4\tau}\right) \right\} \Delta t(\tau) + \frac{\lambda_1}{2 + \pi a_1} \int_0^{\tau} \frac{[\Delta t(\tau) - \Delta t(\xi)]}{(\tau - \xi)^3} \times \left\{ 1 - \sum_{n=1}^{\infty} \left[ 1 + \left(\frac{1 - \varkappa}{\varkappa + 1}\right)^n \right] \frac{n^2 A^2 - 2(\tau - \xi)}{2(\tau - \xi)} \exp\left[-\frac{n^2 A^2}{4(\tau - \xi)}\right] \right\} d\xi .$$
(X)

### **APPENDIX 2**

Using the same substitution of (I), (II), we may state that

$$\Delta t(\tau) = \int_{0}^{\tau} \frac{1/a_{1}}{\lambda_{1} + \overline{n} (\varkappa + 1) (\tau - \xi)} \left\{ (1 - \varkappa) \exp\left(-\frac{\delta^{2}}{a_{1} (\tau - \xi)}\right) - \frac{2}{\alpha_{1} (\tau - \xi)} \right\} + (1 + \varkappa) + \sum_{n=1}^{\infty} \left[ (1 - \varkappa) \exp\left(-\frac{\delta^{2}}{a_{1} (\tau - \xi)}\right) + (1 + \varkappa) + \sum_{n=1}^{\infty} \left[ (1 - \varkappa) \exp\left(-\frac{\kappa}{\alpha_{1} (\tau - \xi)}\right) + (1 + \varkappa) \exp\left(-\frac{\kappa}{\alpha_{1} (\tau - \xi)}\right) - 2 \exp\left[-\frac{(2n + 1)^{2}\delta^{2}}{4a_{1} (\tau - \xi)}\right] \right] \left(\frac{\varkappa - 1}{1 + \varkappa}\right)^{n} \right\} q(\xi) d\xi.$$
(XI)

#### NOTATION

 $q(\tau)$ , non-steady-state heat flux through the heat meter;  $\lambda_1$ ,  $a_1$ ,  $\delta$ , thermal conductivity, thermal diffusivity, and thickness of the heat meter, respectively;  $\lambda_2$ ,  $a_2$ , thermal conductivity and thermal diffusivity, respectively, of the base of the heat meter;  $\Delta t(\tau)$ , temperature gradient over the thickness of the heat meter;  $\eta$ , index of thermal inertia;  $\tau$ , time; s, parameter of Laplace transform;  $t_1(x, \tau)$ , temperature of the heat meter at point x;  $t_2(x, \tau)$ , temperature of the base;  $t_c$ , ambient temperature;  $Y_q(s)$ , transfer function from the heat flux  $q(\tau)$  to the temperature gradient  $\Delta t(\tau)$ .

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# ACCURACY OF TEMPERATURE MEASUREMENTS IN DETERMINING THE THERMAL CONDUCTIVITY OF SUBSTANCES BY STATIONARY METHODS IN THE RANGE OF MODERATE TEMPERATURES

#### O. A. Sergeev

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An empirical comparison is made of the accuracy of platinum-rhodium-platinum and Chromel-Alumel thermocouples in determining the thermal conductivity of substances.

The main errors present in determinations of thermal conductivity are connected with temperature measurements. The most important of these errors, in turn, is that arising in the measurement of the temperature difference  $\vartheta$ , which is necessary to determine the temperature gradient in the specimen. This difference is three-dimensional [1] and, when measured with a differential thermocouple, is calculated as the ratio of the readings of the thermocouple  $\Delta y$  to its sensitivity  $\beta$ . These values are measured with a high degree of accuracy by modern electrical instruments. The low accuracy in the measurement of  $\vartheta$  is connected [1] with the process of determining  $\beta$ . One of the main reasons for this is that, in a given thermophysical experiment, the thermocouple may be used under conditions which differ sharply from those under which it was calibrated on special units in the thermometric laboratory. In particular, this difference leads to a situation whereby the nonuniformity of the thermoelectrodes in these two cases is manifest in different temperature fields and is a source of unknown (with respect to both sign and magnitude) additional emf's in the thermocouple circuit.

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